A Mobius Transformation Sends Circles onto Circles

First, use a Cross-Ratio map to send the circle to the real line.

In[1]:=

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crossRatio[x1_, y1_, x2_, y2_, x3_, y3_, x4_, y4_] :=
   (((x1 + I * y1) - (x3 + I * y3)) * ((x2 + I * y2) - (x4 + I * y4))) /
    (((x2 + I * y2) - (x3 + I * y3)) * ((x1 + I * y1) - (x4 + I * y4)));
  (*Cross Ratio is a special case of a Mobius Transformation*)



Manipulate[
  (*Locating 4 points on the unit circle*)
  θ = π / 2;
  pointsPolar = Table[E^(I*(θ+i)), {i, 1, 3}];
  PrependTo[pointsPolar, E^(I*angle)];
  points = Table[{Re[pointsPolar[[i]]], Im[pointsPolar[[i]]]}, {i, 1, 4}];
  (*Texts to label each point*)
  texts = Table[Text["z"_i, points[[i]] + {0.1, 0.1}], {i, 1, 4}];
  texts2 = Table[Text[{i, 0}, {i, 0} + {1.5, -0.2}], {i, 0, 1}];
  (*Arrows connecting points on unit circle*)
  arrows = Table[Arrow[BezierCurve[{points[[i]], {Re[pointsPolar[[i]] * E^(I* 2*π/16)],
         Im[pointsPolar[[i]] * E^(I* 2*π/16)]}, points[[i + 1]]}]], {i, 1, 3}];
  (*The image of the 3 defining points under the cross ratio. Notice
   that one is sent to the origin, the other to 1, and the last to infinity.*)
  cr = Table[{Re[crossRatio[points[[i, 1]], points[[i, 2]], points[[2, 1]], points[[
          2, 2]], points[[3, 1]], points[[3, 2]], points[[4, 1]], points[[4, 2]]]],
      Im[crossRatio[points[[i, 1]], points[[i, 2]], points[[2, 1]],
        points[[2, 2]], points[[3, 1]], points[[3, 2]], points[[4, 1]],
        points[[4, 2]]]]}, {i, 2, 3}] + {{1.5, 0}, {1.5, 0}};

  (*These images are translated to the right 1.5
   units to make the demonstration more visually appealing*)

  (*The image of the Manipulat-ed point under the Cross Ratio map.*)
  cr1 =
   {Re[crossRatio[points[[1, 1]], points[[1, 2]], points[[2, 1]], points[[2, 2]],
       points[[3, 1]], points[[3, 2]], points[[4, 1]], points[[4, 2]]]],
     Im[crossRatio[points[[1, 1]], points[[1, 2]], points[[2, 1]], points[[2, 2]],
       points[[3, 1]], points[[3, 2]], points[[4, 1]], points[[4, 2]]]]} + {1.5, 0};
  (*Arrows connecting the points, showing the orientation.*)
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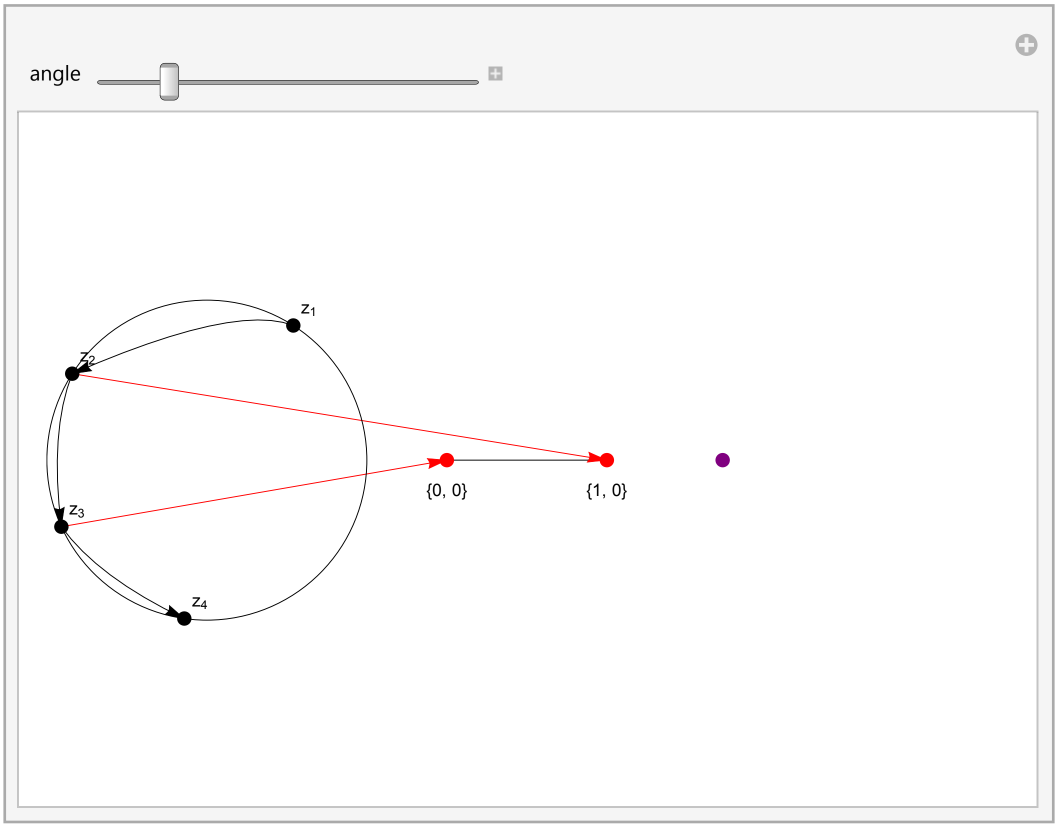
```
lines = Arrow[{{points[[2]], cr[[1]]}, {points[[3]], cr[[2]]}}];
lineSegment = Line[{{1.5, 0}, {2.5, 0}}];
Graphics[{lineSegment, Arrowheads[0.02], Red, lines, Black, arrows, texts,
  texts2, Circle[], PointSize[0.015], Point[points], Red, Point[{cr}],
  Purple, Point[cr1]}, PlotRange → {{-1, 5}, {-2, 2}}, ImageSize → 500]

, {{angle, 1}, .01, 2 * π}]
```

Out[2]=



Then use the inverse of the Cross-Ratio to send the real line to the circle. Cross-Ratios and their inverses are Mobius Transformations.

In[3]:=

```
crossRatio[x1_, y1_, x2_, y2_, x3_, y3_, x4_, y4_] :=
  (((x1 + I * y1) - (x3 + I * y3)) * ((x2 + I * y2) - (x4 + I * y4))) /
   (((x2 + I * y2) - (x3 + I * y3)) * ((x1 + I * y1) - (x4 + I * y4)));
(*The inverse of the cross ratio was retireved using
 Solve[] and plugging in.*)
invCrossRatio[x1_, y1_, x2_, y2_, x3_, y3_, x4_, y4_] :=
  (-x2 x3 + x1 x2 x4 + x3 x4 - x1 x3 x4 + i x2 x4 y1 - i x3 x4 y1 - i x3 y2 + i x1 x4 y2 -
     x4 y1 y2 - i x2 y3 + i x4 y3 - i x1 x4 y3 + x4 y1 y3 + y2 y3 + i x1 x2 y4 + i x3 y4 -
     i x1 x3 y4 - x2 y1 y4 + x3 y1 y4 - x1 y2 y4 - i y1 y2 y4 - y3 y4 + x1 y3 y4 + i y1 y3 y4) /
```

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(-x2 + x1 x2 - x1 x3 + x4 + ⅈ x2 y1 - ⅈ x3 y1 - ⅈ y2 + ⅈ x1 y2 - y1 y2 - ⅈ x1 y3 + y1 y3 + ⅈ y4);


Manipulate[
 (*Locating 4 points on the unit circle*)
 θ = π/2;
 pointsPolar = Table[E^(I*(θ+i)), {i, 1, 3}];
 PrependTo[pointsPolar, E^(I*angle)];
 points = Table[{Re[pointsPolar[[i]]], Im[pointsPolar[[i]]]}, {i, 1, 4}];
 (*Texts to label each point*)
 texts = Table[Text["z"_i, points[[i]] + {0.2, 0.2}], {i, 1, 4}];
 texts2 = Table[Text[{i, 0}, {i, 0} + {1.5, -0.2}], {i, 0, 1}];

 (*Arrows connecting points on unit circle*)
 arrows = Table[Arrow[BezierCurve[{points[[i]], {Re[pointsPolar[[i]] * E^(I*(2*π)/16)],
       Im[pointsPolar[[i]] * E^(I*(2*π)/16)]}, points[[i + 1]]}]], {i, 1, 3}];
 (*The image of the 3 defining points under the cross ratio. Notice
  that one is sent to the origin, the other to 1, and the last to infinity.*)
 (*These images are translated to the right 1.5 units to
  make the demonstration more visually appealing*)
 cr = Table[{Re[crossRatio[points[[i, 1]], points[[i, 2]], points[[2, 1]], points[[
        2, 2]], points[[3, 1]], points[[3, 2]], points[[4, 1]], points[[4, 2]]]],
      Im[crossRatio[points[[i, 1]], points[[i, 2]], points[[2, 1]],
        points[[2, 2]], points[[3, 1]], points[[3, 2]], points[[4, 1]],
        points[[4, 2]]]]}, {i, 2, 3}] + {{1.5, 0}, {1.5, 0}};
 (*The image of the Manipulat-ed point under the Cross Ratio map.*)
 cr1 =
  {Re[crossRatio[points[[1, 1]], points[[1, 2]], points[[2, 1]], points[[2, 2]],
      points[[3, 1]], points[[3, 2]], points[[4, 1]], points[[4, 2]]]],
    Im[crossRatio[points[[1, 1]], points[[1, 2]], points[[2, 1]], points[[2, 2]],
      points[[3, 1]], points[[3, 2]], points[[4, 1]], points[[4, 2]]]]} + {1.5, 0};

 (*Arrows connecting the points, showing the orientation.*)
 lines = Arrow[{{points[[2]], cr[[1]]}, {points[[3]], cr[[2]]}}];
 lineSegment = Line[{{-50, 0}, {50, 0}}];
 lines2 = Arrow[{{{0, 0} + {1.5, 0}, invOfOrigin}, {{1, 0} + {1.5, 0}, invOfUnit}}];
 connectingLine = Arrow[{{points[[1]], cr1}, {cr1, invCr1}}];

 (*Now this code draws the next circle, as the inverse of the
  Cross Ratio of the real line maps to points on the circle.*)
 (*Everything's been translated 4 points to the right
  for the sake of the demonstration of the composition
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 of the two consecutive maps from left to right.*)
points2 = points;
(*The inverse of the Manipulat-
 ed point from the real line back to the unit circle*)
invCr1 = {Re[invCrossRatio[cr1[[1]], cr1[[2]], points2[[2, 1]], points2[[2, 2]],
      points2[[3, 1]], points2[[3, 2]], points2[[4, 1]], points2[[4, 2]]]],
    Im[invCrossRatio[cr1[[1]], cr1[[2]], points2[[2, 1]],
      points2[[2, 2]], points2[[3, 1]], points2[[3, 2]],
      points2[[4, 1]], points2[[4, 2]]]]} + {4, 0};

(*The inverse cross ratio of the origin
 and the unit are sent back onto the unit circle*)
invOfOrigin = {Re[invCrossRatio[0, 0, points2[[2, 1]], points2[[2, 2]],
      points2[[3, 1]], points2[[3, 2]], points2[[4, 1]], points2[[4, 2]]]],
    Im[invCrossRatio[0, 0, points2[[2, 1]], points2[[2, 2]], points2[[3, 1]],
      points2[[3, 2]], points2[[4, 1]], points2[[4, 2]]]]} + {4, 0};
invOfUnit = {Re[invCrossRatio[1, 0, points2[[2, 1]], points2[[2, 2]],
      points2[[3, 1]], points2[[3, 2]], points2[[4, 1]], points2[[4, 2]]]],
    Im[invCrossRatio[1, 0, points2[[2, 1]], points2[[2, 2]], points2[[3, 1]],
      points2[[3, 2]], points2[[4, 1]], points2[[4, 2]]]]} + {4, 0};
(*The circle to the far right*)
circle = Circle[{4, 0}, 1];

text3 = Text["w3", invOfOrigin + {0.15, 0.15}];
text4 = Text["w2", invOfUnit + {0.15, -0.15}];
text5 = Text["w1", invCr1 + {0.2, 0.2}];
arrow2 = Arrow[BezierCurve[{invOfUnit, {2.8, 0}, invOfOrigin}]];

Graphics[{Arrowheads[0.01], circle, Circle[], lineSegment, arrows, texts, texts2,
   text3, text4, text5, Red, lines, PointSize[0.006], Black, Point[points],
   Red, Point[{cr}], Purple, arrow2, lines2, Black, Point[invCr1],
   Point[{invOfOrigin, invOfUnit}], Blue, Point[cr1], Blue, connectingLine},
  PlotRange → {{-7, 10}, {-3, 3}}, ImageSize → 1200]

, {{angle, 1, "Point z1"}, .01, 4 * π}]
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Out[5]=